

The role of the nature of the noise in the thermal conductance of mechanical systems

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Focussing on a paradigmatic small system consisting of two coupled damped oscillators, we survey the role of the Lévy-Itô nature of the noise in the thermal conductance. For white noises, we prove that the Lévy-Itô composition (Lebesgue measure) of the noise is irrelevant for the thermal conductance of a non-equilibrium linearly coupled chain, which signals the independence between mechanical and thermodynamical properties. On the other hand, for the non-linearly coupled case, the two types of properties mix and the explicit definition of the noise plays a central role.

I. INTRODUCTION

The law of heat conduction, or Fourier's law, *i.e.*, the property by which the heat flux density is equal to the product of the thermal conductivity by the negative temperature gradient [1] is a paradigmatic manifestation of the ubiquitous laws of thermodynamics [2]. Recently, it has stoked a significant amount of work on its explicit derivation for large Hamiltonian systems [3–5]. In this context, models with anharmonic coupling succeed in diffusing energy, but the analytic solutions thereto are very demanding, even for the few cases where that is possible. Since they allow a larger number of exactly solvable cases, small systems are worthwhile [6] and particularly relevant in chemical physics and nanosystems [7]. In the scope of analytical methods, we highlight the time averaging of observables that endure a stationary state [8, 9]. This account has several advantages, namely compared with the Fokker-Planck approach, which cannot be applied to those cases where higher than second order cumulants of the noise are non-vanishing and also significant. This comprises Poissonian [10, 13, 14] and other non-Gaussian massive particles [11] as well as other cases where the interaction with a reservoir is described by a process with a non-zero singular part of the measure when a Lévy-Itô (LI) decomposition is applied [12].

Stemming from these facts, we perform a time averaging study of a small non-equilibrium system composed of two damped coupled oscillators at distinct temperatures and determine the explicit formula of the Fourier's law for linear and non-linear cases. In spite of its simplicity, the former has relevant traits: *i)* it is a non-equilibrium system; *ii)* Its heat flux definition is well known; *iii)* It is adjustable to different kinds of reservoirs; *iv)* It can be expanded into a infinite chain with a nearly direct appli-

cation of the results of a $N = 2$ block; *v)* It represents the result of Langevin coloured noises by a renormalisation of the masses [8] and *vi)* Linearity is still a source of important results in many areas [15–18].

II. MODEL

Our problem focus on solving the set of equations,

$$m \frac{dv_i(t)}{dt} = -k x_i(t) - \gamma v_i(t) - \sum_{l=1}^2 k_{2l-1} [x_i(t) - x_j(t)]^{2l-1} + \eta_i(t) \quad (1)$$

with $v_i(t) \equiv \frac{dx_i(t)}{dt}$, where $(i, j) \in \{1, 2\}$ and k_1 and k_3 are the linear and non-linear coupling constants, respectively. The system is decoupled (linear) for $k_{1(3)} = 0$. The transfer flux, $j_{12}(t)$, between the two particles reads,

$$j_{12}(t) \equiv - \sum_{l=1}^2 \frac{k_{2l-1}}{2} [x_1(t) - x_2(t)]^{2l-1} [v_1(t) + v_2(t)]. \quad (2)$$

The term, $\eta_i(t)$, represents a general uncorrelated Lévy class stochastic process with cumulants,

$$\langle \eta_{i_1}(t_1) \dots \eta_{i_n}(t_n) \rangle_c = \mathcal{A}(t_1, n) \delta_{i_1 i_2} \dots \delta_{i_{n-1} i_n} \times \delta(t_1 - t_2) \dots \delta(t_{n-1} - t_n). \quad (3)$$

From [19], we either have two or infinite non-zero cumulants. The former corresponds to the case in which the measure is absolutely continuous, characterising a Brownian process. In Eq. (3), $\mathcal{A}(t, n)$ is described by the noise; If it is Wiener-like, $W(t) \equiv \int_{t_0}^t \eta(t') dt'$, $\mathcal{A}(t, n)$ is time-independent and equal to σ^2 for $n = 2$ and zero otherwise (σ is the standard deviation of the Gaussian). Among infinite non-zero cumulant noises, we can include the Poisson process for which $\mathcal{A}(t, n)$ equals $\overline{\Phi^n} \lambda(t)$ [20], with Φ being the $p(\Phi)$ independent and identically distributed magnitude and $\lambda(t)$ the rate of shots. Herein, \mathcal{A} is time-independent without loss of generality. For

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$k_1 = k_3 = 0$, Eq. (1) is totally decoupled and the solutions to the problem of homogeneous and sinusoidal heterogeneous Poisson noises can be found in Ref. [10].

III. RESULTS

Laplace transforming $x_i(t)$ and $v_i(t)$ we obtain,

$$\tilde{x}_i(s) = \frac{k_1}{R(s)} \tilde{x}_j(s) + \frac{\tilde{\eta}_i(s)}{R(s)} + \frac{k_2}{R(s)} \lim_{\alpha \rightarrow 0} \iiint \frac{\prod_{n=1}^3 \frac{dq_n}{2\pi} [\tilde{x}_i(i q_n + \alpha) - \tilde{x}_j(i q_n + \alpha)]}{s - (i q_1 + i q_2 + i q_3 + 3\alpha)}, \quad (4)$$

$$s \tilde{x}_i(s) = \tilde{v}_i(s)$$

($\text{Re}(s) > 0$) with $R(s) \equiv (m s^2 + \gamma s + k + k_1)$. The solutions to Eq. (4) are obtained considering the relative position, $\tilde{r}_D(s) \equiv \tilde{x}_1(s) - \tilde{x}_2(s)$, the mid-point position, $\tilde{r}_S(s) \equiv (\tilde{x}_1(s) + \tilde{x}_2(s))/2$, as well as the respective noises $\tilde{\eta}_D(s) \equiv \tilde{\eta}_1(s) - \tilde{\eta}_2(s)$ and $\tilde{\eta}_S(s) \equiv (\tilde{\eta}_1(s) + \tilde{\eta}_2(s))/2$. After some algebra it yields,

$$\begin{cases} \tilde{r}_D(s) = \frac{\tilde{\eta}_D(s)}{R'(s)} - \frac{2k_3}{R'(s)} \lim_{\alpha \rightarrow 0} \int_{-\infty}^{+\infty} \frac{\prod_{l=1}^3 \frac{dq_l}{2\pi} \tilde{r}_D(i q_l + \alpha)}{s - \sum_{l=1}^3 (i q_l + \alpha)}, \\ \tilde{r}_S = \frac{\tilde{\eta}_S(s)}{R''(s)} \end{cases}, \quad (5)$$

with $R'(s) \equiv (m s^2 + \gamma s + k + 2k_1)$ and $R''(s) \equiv (m s^2 + \gamma s + k)$. Reverting Eq. 5 we get $\tilde{x}_1(s)$ and

$\tilde{x}_2(s)$. Concomitantly, we must compute the Laplace transforms of η_1 and η_2 ,

$$\begin{aligned} \langle \tilde{\eta}_{i_1}(z_1) \dots \tilde{\eta}_{i_n}(z_n) \rangle_c &= \int_0^\infty \prod_{j=1}^n dt_{i_j} \exp \left[- \sum_{j=1}^n z_{i_j} t_{i_j} \right] \\ &\quad \times \langle \eta_{i_1}(t_1) \dots \eta_{i_n}(t_n) \rangle_c \\ &= \frac{\mathcal{A}(n)}{\sum_{j=1}^n z_{i_j}} \delta_{i_1 i_2} \dots \delta_{i_{n-1} i_n}. \end{aligned} \quad (6)$$

that are employed in the averages over time [8],

$$\overline{\langle x_a^m v_b^n \rangle_c} = \lim_{z \rightarrow 0} z \int \int \int \delta(t - t_1) \delta(t - t_2) e^{-z t} \langle x_a^m(t_1) v_b^n(t_2) \rangle_c dt_1 dt_2 dt, \quad (7)$$

$$= \lim_{z, \varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \prod_{l=1}^{m+n} \frac{dq_l}{2\pi} z \frac{\prod_{l'=1}^n (i q_{l'} + \varepsilon) \langle \prod_{l=1}^{m+n} \tilde{x}(i q_l + \varepsilon) \rangle_c}{z - (\sum_{l'=1}^{m+n} i q_{l'} + (m+n)\varepsilon)}. \quad (8)$$

Allowing for a contour that goes along the straight line from $-\rho + i\varepsilon$ to $\rho + i\varepsilon$ and then counterclockwise along a semicircle centered at $0 + i\varepsilon$ from $\rho + i\varepsilon$ to $-\rho + i\varepsilon$ ($\rho \rightarrow \infty$ and $\varepsilon \rightarrow 0$), we realise that two situations occur: either the calculation of the residues leads us to a term like $\frac{z}{z-w} u$, with $(u, w) \neq 0$, which vanishes in the limit $z \rightarrow 0$, or to $\frac{z}{z} u$, which is non-zero. The problem solving now resumes to an expansion of Eq. (2) in powers of k_3 , actually the expansion is carried out in powers of $k_3 T/k_1^2$ [21] (see Appendices). Hereinafter, besides the results of the Brownian thermostats, which corresponds to the pure continuous measure, we make explicit the conductance for the Poisson reservoirs, the epitome of singular measures. Nevertheless, the results for other non-Gaussian noises can be obtained following our methodology yielding the same qualitative results. In

first order the transfer flux reads,

$$\overline{\langle j_{12} \rangle} = \overline{\langle j_{12}^{(0)} \rangle} + \overline{\langle j_{12}^{(1)} \rangle} + \overline{\langle j_{12}^{(s)} \rangle} + \mathcal{O}(k_3^2), \quad (9)$$

with,

$$\begin{cases} \overline{\langle j_{12}^{(0)} \rangle} = -\frac{k_1^2}{4} \frac{[\mathcal{A}_1(2) - \mathcal{A}_2(2)]}{m k_1^2 + \gamma^2(k + k_1)} \\ \overline{\langle j_{12}^{(1)} \rangle} = -\frac{3}{8} \gamma k_1 k_3 \frac{(2k + k_1)[\mathcal{A}_1(2)^2 - \mathcal{A}_2(2)^2]}{(k + 2k_1)[\gamma^2(k + k_1) + m k_1^2]^2} \end{cases}, \quad (10)$$

and

$$\overline{\langle j_{12}^{(s)} \rangle} = -\frac{27}{2} \gamma^2 \frac{k_1 k_3}{\lambda} \frac{\mathcal{N}}{\mathcal{D}} ([\mathcal{A}_1(2)^2 - \mathcal{A}_2(2)^2]), \quad (11)$$

for the Poisson case and $\overline{\langle j_{12}^{(s)} \rangle} = 0$ for the Brownian case (see Appendices). For Poisson, when $\lambda^{-1} \ll 1$ (keeping

the temperature fixed), the weight of the singularity of the noise measure dwindles and $\overline{\langle j_{12}^{(s)} \rangle} \rightarrow 0$. The coefficients in Eq. (11) are,

$$\mathcal{N} \equiv \gamma^2 (5k + 3k_1) + m (3k_1^2 + 4k^2 + 11kk_1), \quad (12)$$

$$\mathcal{D} \equiv \left[m(4k + 9k_1)^2 + 6\gamma^2(2k + 3k_1) \right] \times \quad (13)$$

$$\left[3\gamma^4 + m^2k_1^2 + 4m\gamma^2(k + k_1) \right] [\gamma^2(k + k_1)].$$

Thence, we are finally in the position to compute the thermal conductance,

$$\kappa \equiv -\frac{\partial}{\partial \Delta T} \langle j_{12} \rangle_{\Delta T}, \quad (14)$$

$$\kappa = -\frac{\overline{\langle j_{12} \rangle}}{T_1 - T_2}.$$

Resorting to single particle results and the equipartition theorem [10], we relate the cumulants of the noise and the proper temperature, T_i , namely, $\mathcal{A}_i(2) = 2\gamma T_i$, yielding a thermal conductance, $\kappa = \kappa^{(0)} + \kappa^{(1)} + \kappa^{(s)} + \mathcal{O}(k_3^3)$. Equations (9)-(11) pave the way to the following assertion; *When interacting particles are subject to white reservoirs and coupled in a linear form, the explicit thermal conductance is independent of the specific nature of the noise, namely the outcome of their Lévy-Itô decomposition, whereas for non-linear coupling the nature of the measure of the noise (its decomposition) is pivotal.* In other words, the linear case is heedless of the measure of the reservoirs and it only takes into consideration their temperatures. Hence, $\kappa^{(0)}$ is exactly the same, either we have a Wiener noise (continuous measure), which is the standard noise in fundamental statistical mechanics studies [9, 22], or a Poisson noise (paradigmatic case of singular measure). Although the linear coupling result has only been explicitly proved for two particles, it is valid for general N . In fact, *for a linear chain, the local energy flow $\langle j_{i,i+1} \rangle$ can be written as a function of the cumulants $\langle \eta_i(z)\eta_1(z') \rangle_c = 2\gamma T_i$ and $\langle \eta_{i+1}(z)\eta_{i+1}(z') \rangle_c = 2\gamma T_{i+1}$, wherein the dependence on the specific nature of the noise is eliminated, except for the respective temperatures.* On the other hand, if the nature of the noise affects the conductance of the simplest coupling element, the conductance for generic chains is also changed. This result is unexpected since contrarily to single particle linear cases, wherein the LI nature is already relevant [10, 11], for coupled systems the LI composition does solely become significant when the interaction between the elements of the system happens in a non-linear way. Only in this case higher-order cumulants of the noise, which can be learnt as higher-order sources of energy, influence the result. For the same (k_1, k_3) , in decreasing the singularity by soaring λ , the two thermal conductances tally.

To further illustrate these results, we have simulated cases of equally massive particles subject to Wiener and Poisson noises at different temperatures, T . For the former, we have $T = \sigma^2/2$ whereas for the latter, we have assumed a homogeneous Poisson process with a

rate of events λ , with a random amplitude, Φ , exponentially distributed, $p(\Phi) \sim \exp[-\Phi/\bar{\Phi}]$, which yields $T = \lambda_0 \bar{\Phi}^2/\gamma$ [10]. In Fig. 1, we depict linear coupling. It is visible that after a transient time, t^* , the system reaches a stationary state and $\langle j_{12} \rangle$ becomes equal to $\overline{\langle j_{12} \rangle}$, whatever the reservoirs. In fact, even more complex models, such as linear chains of oscillators, verify the $\kappa = \kappa^{(0)}$ property. Still, this is valid when each particle is perturbed by different types of noise, *e.g.*, a Brownian particle coupled with a Poissonian particle. The instance where the noises are of different nature gives rise to an apparent larger value of the standard deviation.¹

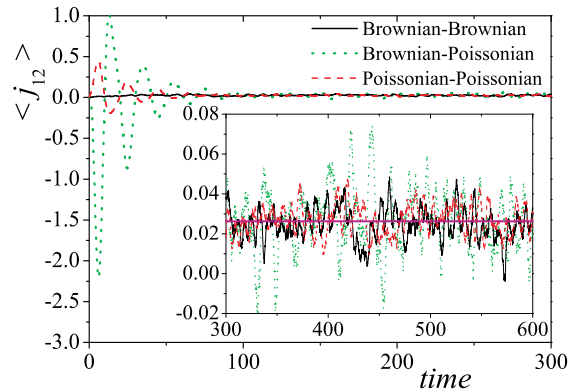


FIG. 1. (Colour on-line) Average exchange flux $\langle j_{12} \rangle$ of a two massive particle system for different combinations of paradigmatic types of noise with $T_1 = 10$ and $T_2 = 121/10$, $m = 10$, $\gamma = k = 1$, $k_1 = 1/5$, $k_3 = 0$ and $\lambda = 10$ for Poissonian particles. After the transient, κ agrees with the theoretical value, $\kappa = 21/800 = 0.02625$, with the fitting curves lying within line thickness. The averages have been obtained averaging over $850 \times (5 \times 10^5)$ points. The discretisation used is $\delta t = 10^{-5}$ with snapshots at every $\Delta t = 10^{-3}$.

In turning $k_3 \neq 0$, the composition of the measure of the reservoirs comes into play. In Fig. 2, we show the difference between equivalent Brownian and Poissonian particles with a good agreement between the averages over numerical realisations and the respective (first order) approximation. For the same temperature, the larger the value of k , the larger the value of k_3^* defining a 10% difference between numerical values the approximation.

IV. FINAL REMARKS

To summarise, we have studied the thermal conduction in a paradigmatic mechanical system composed of two coupled damped harmonic oscillators subject to generic

¹ Although the computation of $\sigma_{j_{12}}^2 \equiv \langle j_{12}^2 \rangle - \langle j_{12} \rangle^2$ is possible, we have set it by as it demands a mathematical *tour de force* likely to yield a lengthy formula with little grasping information.

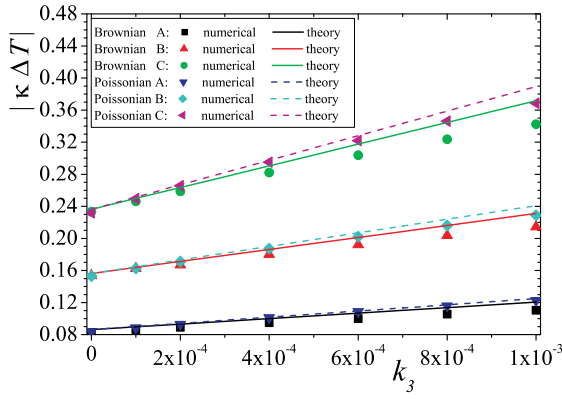


FIG. 2. (Colour on-line) Comparison between numerically obtained values (symbols) and the first order approximation of thermal conductance from Eqs. (9)-(11) for different temperatures pairs, namely $A = \{10, \frac{169}{10}\}$, $B = \{10, \frac{225}{10}\}$, $C = \{10, \frac{289}{10}\}$ with $m = 10$, $\gamma = k = 1$, $k_1 = 1/5$ and $\lambda = 1$ for Poissonian particles.

noises, which can be understood as a concise way to describe non-equilibrium problems. By averaging in the Laplace space, we have been able to determine the conductance of a linearly coupled system and approximate formulae for non-linearly coupled particles. We have shown the conductance of the former is independent of the nature of the (white) noise, namely its Lévy-Itô decomposition structure. This result is unexpected since the measure of the thermal bath plays a major role for single particle properties. The dependence on the noise only emerges when there exists transfer of energy in a non-linear way and higher-order cumulants of the noise enter in the calculations. In the case of Poisson noises, we show that the difference to Brownian noises becomes negligible when the ratio between the coupling constants and the rate of events is small. Our calculations evidence

the independence of the thermodynamical properties of the system from the nature of the reservoirs in linearly coupled systems. On the other hand, when the coupling is non-linear, the nature of the reservoirs affects the conductance, which represents a mixture between mechanic and thermodynamical properties of the system.

Our results have direct implication for the study of the thermal conductance of systems under the influence of noises other than Wiener, for instance: *i)* solid state problems wherein shot (singular measure) noise is related to the quantisation of the charge [23]; *ii)* RLC circuits with injection of power at some rate resembling heat pumps [13]; *iii)* Surface diffusion and low vibrational motion with adsorbates, *e.g.*, Na/Cu(001) compounds [25]; *iv)* Biological motors in which shot noise mimics the nonequilibrium stochastic hydrolysis of adenosine triphosphate [14] and *v)* Molecular dynamics when the Andersen thermostat is applied. Actually, in molecular dynamics [24], the Langevin reservoir is just one in a large collection of baths represented by our definition of noise (3). In these problems, for non-linearly coupled elements the experimentally measured energy flux will be greater than the energy flux given by Langevin reservoirs at the same temperatures and equal if coupling is linear. At the theoretical level, the method is worth being used to shed light on non-linear chains as well. Within this context, the feasible approach is once again to consider a perturbative expansion of the non-linearities in the problem.

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Appendix A: Heat flux definition

As indicated in the main document the model consists of two masses connected by linear and non-linear springs so that,

$$\begin{aligned} \dot{x}_1 &= v_1, \\ \dot{x}_2 &= v_2, \\ m\ddot{x}_1 &= -kx_1 - k_1(x_1 - x_2) - k_3(x_1 - x_2)^3 - \gamma\dot{x}_1 + \eta_1, \\ m\ddot{x}_2 &= -kx_2 - k_1(x_2 - x_1) - k_3(x_2 - x_1)^3 - \gamma\dot{x}_2 + \eta_2. \end{aligned} \quad (A1)$$

The flux of energy $J_{1 \rightarrow 2}$ defines the thermal conductance. For this unidimensional mechanical system, the transmitted power is by the the instantaneous power difference [9]:

$$J_{1 \rightarrow 2} = \frac{dW_{1 \rightarrow 2} - dW_{2 \rightarrow 1}}{2 dt} = \frac{F_{1 \rightarrow 2}v_2 - F_{2 \rightarrow 1}v_1}{2}, \quad (A2)$$

where $F_{\alpha \rightarrow \beta}$ is the force exerted on particle β by particle α .

Assuming a conservative potential, the form of the force between the particles will be,

$$F_{\alpha \rightarrow \beta} = -k_1(x_\alpha - x_\beta) - k_3(x_\alpha - x_\beta)^3. \quad (\text{A3})$$

Our goal is to derive a systematic expansion for the time-average $j_{12} \equiv \langle J_{1 \rightarrow 2} \rangle \equiv \kappa (T_2 - T_1)$, where κ is the thermal conductance of the model. Thus, we can write

$$\begin{aligned} j_{12} &= \left\langle \frac{F_{1 \rightarrow 2} v_2 - F_{2 \rightarrow 1} v_1}{2} \right\rangle \\ &= -k_1 \left\langle \frac{x_1 v_2 - x_2 v_1}{2} \right\rangle - k_3 \left\langle \frac{(x_1 - x_2)^3 (v_2 + v_1)}{2} \right\rangle, \end{aligned} \quad (\text{A4})$$

where the terms of the form $\langle x_1^n v_1 \rangle$ and $\langle x_2^n v_2 \rangle$ vanish identically as $t \rightarrow \infty$ [9].

1. The Gaussian case

The noise functions are assumed to be white and Gaussian,

$$\begin{aligned} \langle \eta_1(t_1) \rangle_c &= 0, \\ \langle \eta_1(t_1) \eta_1(t_2) \rangle_c &= 2\gamma T_1 \delta(t_1 - t_2), \\ \langle \eta_2(t_1) \rangle_c &= 0, \\ \langle \eta_2(t_1) \eta_2(t_2) \rangle_c &= 2\gamma T_2 \delta(t_1 - t_2), \end{aligned} \quad (\text{A5})$$

The initial conditions will be assumed to be

$$x_1(0) = v_1(0) = x_2(0) = v_2(0) = 0. \quad (\text{A6})$$

The Laplace transforms read,

$$\begin{aligned} \tilde{v}_1(s) &= s\tilde{x}_1(s), \\ \tilde{v}_2(s) &= s\tilde{x}_2(s), \end{aligned} \quad (\text{A7})$$

using $R(s) \equiv (m s^2 + \gamma s + (k + k_1)) = m(s^2 + \theta s + \omega^2) = (s - \zeta_+)(s - \zeta_-)$, where

$$\zeta_{\pm} = -\frac{\theta}{2} \pm \frac{i}{2} \sqrt{4\omega^2 - \theta^2} \quad (\text{A8})$$

we have,

$$\begin{aligned} \tilde{x}_1(s) &= \frac{k_1}{R(s)} \tilde{x}_2(s) + \frac{\tilde{\eta}_1(s)}{R(s)} - \frac{k_3}{R(s)} \lim_{\alpha \rightarrow 0} \int_{-\infty}^{+\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{+\infty} \frac{dq_2}{2\pi} \int_{-\infty}^{+\infty} \frac{dq_3}{2\pi} \times \\ &\times \frac{(\tilde{x}_1(iq_1 + \alpha) - \tilde{x}_2(iq_1 + \alpha)) (\tilde{x}_1(iq_2 + \alpha) - \tilde{x}_2(iq_2 + \alpha)) (\tilde{x}_1(iq_3 + \alpha) - \tilde{x}_2(iq_3 + \alpha))}{s - (iq_1 + iq_2 + iq_3 + 3\alpha)}, \end{aligned} \quad (\text{A9})$$

and

$$\begin{aligned} \tilde{x}_2(s) &= \frac{k_1}{R(s)} \tilde{x}_1(s) + \frac{\tilde{\eta}_2(s)}{R(s)} + \frac{k_3}{R(s)} \lim_{\alpha \rightarrow 0} \int_{-\infty}^{+\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{+\infty} \frac{dq_2}{2\pi} \int_{-\infty}^{+\infty} \frac{dq_3}{2\pi} \times \\ &\times \frac{(\tilde{x}_1(iq_1 + \alpha) - \tilde{x}_2(iq_1 + \alpha)) (\tilde{x}_1(iq_2 + \alpha) - \tilde{x}_2(iq_2 + \alpha)) (\tilde{x}_1(iq_3 + \alpha) - \tilde{x}_2(iq_3 + \alpha))}{s - (iq_1 + iq_2 + iq_3 + 3\alpha)}. \end{aligned} \quad (\text{A10})$$

An straightforward way to write a recurrence equation for the problem above is to take the difference:

$$\begin{aligned} \left(1 + \frac{k_1}{R(s)}\right) (\tilde{x}_1(s) - \tilde{x}_2(s)) &= \frac{\tilde{\eta}_1(s) - \tilde{\eta}_2(s)}{R(s)} - \frac{2k_3}{R(s)} \lim_{\alpha \rightarrow 0} \int_{-\infty}^{+\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{+\infty} \frac{dq_2}{2\pi} \int_{-\infty}^{+\infty} \frac{dq_3}{2\pi} \times \\ &\times \frac{(\tilde{x}_1(iq_1 + \alpha) - \tilde{x}_2(iq_1 + \alpha)) (\tilde{x}_1(iq_2 + \alpha) - \tilde{x}_2(iq_2 + \alpha)) (\tilde{x}_1(iq_3 + \alpha) - \tilde{x}_2(iq_3 + \alpha))}{s - (iq_1 + iq_2 + iq_3 + 3\alpha)}. \end{aligned} \quad (\text{A11})$$

Defining $R'(s) \equiv (m s^2 + \gamma s + (k + 2k_1)) = (s^2 + \theta s + \omega_1^2) = (s - \zeta_{1+})(s - \zeta_{1-})$, where,

$$\zeta_{1\pm} = -\frac{\theta}{2} \pm \frac{i}{2} \sqrt{4\omega_1^2 - \theta^2}$$

we obtain the difference,

$$\begin{aligned}
\tilde{r}_D &= \tilde{x}_1(s) - \tilde{x}_2(s) \\
&= \frac{\tilde{\eta}_1(s) - \tilde{\eta}_2(s)}{R'(s)} - \frac{2k_3}{R'(s)} \lim_{\alpha \rightarrow 0} \int_{-\infty}^{+\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{+\infty} \frac{dq_2}{2\pi} \int_{-\infty}^{+\infty} \frac{dq_3}{2\pi} \times \\
&\quad \times \frac{(\tilde{x}_1(iq_1 + \alpha) - \tilde{x}_2(iq_1 + \alpha)) (\tilde{x}_1(iq_2 + \alpha) - \tilde{x}_2(iq_2 + \alpha)) (\tilde{x}_1(iq_3 + \alpha) - \tilde{x}_2(iq_3 + \alpha))}{s - (iq_1 + iq_2 + iq_3 + 3\alpha)}.
\end{aligned} \tag{A12}$$

Defining $R''(s) \equiv (m s^2 + \gamma s + k) = (s^2 + \theta s + \omega_2^2) = (s - \zeta_{2+})(s - \zeta_{2-})$, where,

$$\zeta_{2\pm} = -\frac{\theta}{2} \pm \frac{i}{2} \sqrt{4\omega_2^2 - \theta^2}$$

we obtain the sum as,

$$\begin{aligned}
\tilde{r}_S &= \frac{\tilde{x}_1(s) + \tilde{x}_2(s)}{2} \\
&= \frac{\tilde{\eta}_1(s) + \tilde{\eta}_2(s)}{2 R''(s)}.
\end{aligned} \tag{A13}$$

Inverting the relations it yields,

$$\begin{aligned}
\tilde{x}_1(s) &= \tilde{r}_S + \frac{\tilde{r}_D}{2}, \\
\tilde{x}_2(s) &= \tilde{r}_S - \frac{\tilde{r}_D}{2}.
\end{aligned} \tag{A14}$$

In the same way, we can define the difference and the average of the noise as,

$$\begin{aligned}
\tilde{\eta}_S(s) &= \frac{\tilde{\eta}_1 + \tilde{\eta}_2}{2}, \\
\tilde{\eta}_D(s) &= \tilde{\eta}_1 - \tilde{\eta}_2.
\end{aligned} \tag{A15}$$

We can now express the recurrence relations for the new variables,

$$\begin{aligned}
\tilde{r}_D(s) &= \frac{\tilde{\eta}_D(s)}{R'(s)} - \frac{2k_3}{R'(s)} \lim_{\alpha \rightarrow 0} \int_{-\infty}^{+\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{+\infty} \frac{dq_2}{2\pi} \int_{-\infty}^{+\infty} \frac{dq_3}{2\pi} \times \\
&\quad \times \frac{\tilde{r}_D(iq_1 + \alpha) \tilde{r}_D(iq_2 + \alpha) \tilde{r}_D(iq_3 + \alpha)}{s - (iq_1 + iq_2 + iq_3 + 3\alpha)}, \\
\tilde{r}_S &= \frac{\tilde{\eta}_S(s)}{R''(s)}.
\end{aligned} \tag{A16}$$

The Laplace transforms of the noise are,

$$\begin{aligned}
\langle \tilde{\eta}_D(s_1) \tilde{\eta}_D(s_2) \rangle_c &= \frac{2\gamma(T_1 + T_2)}{s_1 + s_2}, \\
\langle \tilde{\eta}_S(s_1) \tilde{\eta}_S(s_2) \rangle_c &= \frac{\gamma(T_1 + T_2)}{2(s_1 + s_2)}, \\
\langle \tilde{\eta}_S(s_1) \tilde{\eta}_D(s_2) \rangle_c &= \frac{\gamma(T_1 - T_2)}{s_1 + s_2}.
\end{aligned} \tag{A17}$$

Appendix B: Heat conductance

1. General expression

The series expansion for the thermal flux reads,

$$j_{12} = -\frac{k_1}{2} \langle (x_1 - x_2)(v_2 + v_1) \rangle - \frac{k_3}{2} \langle (x_1 - x_2)^3(v_2 + v_1) \rangle \tag{B1}$$

We can write ($a, b = 1, 2$),

$$\begin{aligned}
\overline{\langle x_a^m v_b^n \rangle_c} &= \lim_{z \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \prod_{k=1}^m \frac{dq_k}{2\pi} \int_{-\infty}^{\infty} \prod_{l=1}^n \frac{dq_l}{2\pi} \frac{z}{z - (\sum_{k=1}^m iq_k + \sum_{l=1}^n iq_l + (m+n)\epsilon)} \langle \prod_{k=1}^m \tilde{x}(iq_k + \epsilon) \prod_{l=1}^n \tilde{v}(iq_l + \epsilon) \rangle_c \\
&= \lim_{z \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \prod_{k=1}^{m+n} \frac{dq_k}{2\pi} \frac{z}{z - (\sum_{k=1}^{m+n} iq_k + (m+n)\epsilon)} \prod_{l=1}^n (iq_l + \epsilon) \left\langle \prod_{k=1}^{m+n} \tilde{x}(iq_k + \epsilon) \right\rangle_c.
\end{aligned} \tag{B2}$$

We can thus express the heat flux $\overline{\langle j_{12} \rangle}$ as,

$$\begin{aligned} \overline{\langle j_{12} \rangle} &= -k_1 \lim_{z \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{\infty} \frac{dq_2}{2\pi} \frac{z}{z-(iq_1+iq_2+2\epsilon)} (iq_2 + \epsilon) \langle \tilde{r}_D(iq_1 + \epsilon) \tilde{r}_S(iq_2 + \epsilon) \rangle_c - \\ &\quad - k_3 \lim_{z \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{\infty} \frac{dq_2}{2\pi} \int_{-\infty}^{\infty} \frac{dq_3}{2\pi} \int_{-\infty}^{\infty} \frac{dq_4}{2\pi} \frac{z}{z-(iq_1+iq_2+iq_3+iq_4+2\epsilon)} \times \\ &\quad \times \{ (iq_4 + \epsilon) \langle \tilde{r}_D(iq_1 + \epsilon) \tilde{r}_D(iq_2 + \epsilon) \tilde{r}_D(iq_3 + \epsilon) \tilde{r}_S(iq_4 + \epsilon) \rangle_c \}. \end{aligned} \quad (B3)$$

We now proceed to expand the heat flow in powers of k_3 .

2. Order zero on k_3

The order zero term is,

$$\begin{aligned} \overline{\langle j_{12} \rangle}_0 &= -k_1 \lim_{z \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{\infty} \frac{dq_2}{2\pi} \frac{z (iq_2 + \epsilon)}{z-(iq_1+iq_2+2\epsilon)} \frac{\langle \tilde{\eta}_D(iq_1 + \epsilon) \tilde{\eta}_S(iq_2 + \epsilon) \rangle_c}{R'(iq_1 + \epsilon) R''(iq_2 + \epsilon)} \\ &= -\frac{k_1}{m^2} \lim_{z \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{\infty} \frac{dq_2}{2\pi} \frac{z (iq_2 + \epsilon)}{z-(iq_1+iq_2+2\epsilon)} \frac{\langle \tilde{\eta}_D(iq_1 + \epsilon) \tilde{\eta}_S(iq_2 + \epsilon) \rangle_c}{R'(iq_1 + \epsilon) R''(iq_2 + \epsilon)} \\ &= -\frac{\gamma (T_1 - T_2) k_1}{m^2} \lim_{z \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{\infty} \frac{dq_2}{2\pi} \frac{z}{z-(iq_1+iq_2+2\epsilon)} \frac{(iq_2 + \epsilon)}{R'(iq_1 + \epsilon) R''(iq_2 + \epsilon)} \frac{1}{(iq_1 + iq_2 + 2\epsilon)} \\ &= -\frac{1}{2} \frac{\gamma (T_1 - T_2) k_1^2}{\gamma^2 (k + k_1) + k_1^2 m}, \end{aligned}$$

which is already a well known result [9, 10].

3. Order 1 on k_3

There are two contributions to order 1 on k_3 : one from the quadratic term and another from the quartic one. The quadratic term reads,

$$\begin{aligned} \overline{\langle j_{12} \rangle}_{12} &= k_1 \lim_{z \rightarrow 0} \lim_{\epsilon \rightarrow 0} \lim_{\alpha \rightarrow 0} \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \cdots \int_{-\infty}^{+\infty} \frac{dq_5}{2\pi} \frac{z (iq_2 + \epsilon)}{z-(iq_1+iq_2+2\epsilon)} \frac{2 k_3}{R'(iq_1 + \epsilon)} \frac{\langle \tilde{r}_D(iq_3 + \alpha) \tilde{r}_D(iq_4 + \alpha) \tilde{r}_D(iq_5 + \alpha) \tilde{r}_S(iq_2 + \epsilon) \rangle_c}{iq_1 + \epsilon - (iq_3 + iq_4 + iq_5 + 3\alpha)} \\ &= 12 \gamma^2 k_1 k_3 \lim_{z \rightarrow 0} \lim_{\epsilon \rightarrow 0} \lim_{\alpha \rightarrow 0} \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \cdots \int_{-\infty}^{+\infty} \frac{dq_5}{2\pi} \frac{z}{z-(iq_1+iq_2+2\epsilon)} \frac{(iq_2 + \epsilon)}{iq_1 + \epsilon - (iq_3 + iq_4 + iq_5 + 3\alpha)} \times \\ &\quad \times \frac{1}{R'(iq_1 + \epsilon) R'(iq_3 + \alpha) R'(iq_4 + \alpha) R'(iq_5 + \alpha) R''(iq_2 + \epsilon)} \frac{(T_1 + T_2) (T_1 - T_2)}{(iq_3 + iq_4 + 2\alpha) (iq_5 + \alpha + iq_2 + \epsilon)} \\ &= -\frac{3}{2} \gamma k_3 \frac{(\gamma^2 k - m k_1^2) (T_1^2 - T_2^2)}{(k + 2 k_1) [m k^2 + \gamma^2 (k + k_1)]^2}, \end{aligned} \quad (B4)$$

The quartic term reads,

$$\begin{aligned} \overline{\langle j_{12} \rangle}_{12} &= -k_3 \lim_{z \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \cdots \int_{-\infty}^{\infty} \frac{dq_4}{2\pi} \frac{z}{z-(iq_1+iq_2+iq_3+iq_4+2\epsilon)} \times \\ &= \times \{ (iq_2 + \epsilon) \langle \tilde{r}_D(iq_1 + \epsilon) \tilde{r}_S(iq_2 + \epsilon) \tilde{r}_D(iq_3 + \epsilon) \tilde{r}_D(iq_4 + \epsilon) \rangle_c \} \\ &= -6 \gamma^2 k_3 \lim_{z \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \cdots \int_{-\infty}^{\infty} \frac{dq_4}{2\pi} \frac{z}{z-(iq_1+iq_2+iq_3+iq_4+2\epsilon)} \times \\ &\quad \times \frac{(iq_2 + \epsilon)}{R'(iq_1 + \epsilon) R'(iq_3 + \alpha) R'(iq_4 + \epsilon) R''(iq_2 + \epsilon)} \frac{(T_1 + T_2) (T_1 - T_2)}{(iq_3 + iq_4 + 2\alpha) (iq_1 + iq_2 + 2\epsilon)} \\ &= -\frac{3}{2} \frac{\gamma k_1 k_3 (T_1^2 - T_2^2)}{(k + 2 k_1) [\gamma^2 (k + k_1) + m k_1^2]}. \end{aligned} \quad (B5)$$

The total term reads,

$$\overline{\langle j_{12}^{(1)} \rangle} = -\frac{3}{2} \frac{\gamma^3 k_1 k_3 (2k + k_1) (T_1^2 - T_2^2)}{(k + 2 k_1) [m k^2 + \gamma^2 (k + k_1)]^2}. \quad (B6)$$

By the relation $j_{12} = -\kappa\Delta T$, we obtain the expansion of the thermal conductance as,

$$\begin{aligned}\kappa &= \frac{\gamma k_1^2}{\gamma^2(k+k_1)+k_1^2 m} + \frac{3}{2} \frac{\gamma^3 k_1 k_3 (2k+k_1)(T_1+T_2)}{(k+2k_1)[mk^2+\gamma^2(k+k_1)]^2} + \mathcal{O}(k_3^2) \\ \kappa &= \kappa^{(0)} + \kappa^{(1)} + \mathcal{O}(k_3^2).\end{aligned}\tag{B7}$$

Appendix C: Poisson Conductance

An interesting use of our formalism is to compare the result of Eq. (B7) with the case of Poisson noise. For Poisson we have [10],

$$\begin{aligned}\langle \tilde{\eta}(s_1)\tilde{\eta}(s_2) \rangle_c &= \frac{2\gamma T}{s_1+s_2}, \\ \langle \tilde{\eta}(s_1)\tilde{\eta}(s_2)\tilde{\eta}(s_3)\tilde{\eta}(s_4) \rangle_c &= \frac{1}{\lambda} \frac{24\gamma^2 T^2}{s_1+s_2+s_3+s_4},\end{aligned}\tag{C1}$$

where the expression for the fourth order cumulant can be checked by dimensional analysis and compared with that of reference [10].

The combinations $\langle \tilde{\eta}_{S,D}\tilde{\eta}_{S,D} \rangle$ need to be reexamined. For the Poisson noise, they yield,

$$\begin{aligned}\langle \tilde{\eta}_D(s_1)\tilde{\eta}_D(s_2) \rangle_c &= \frac{2\gamma(T_1+T_2)}{s_1+s_2}, \\ \langle \tilde{\eta}_S(s_1)\tilde{\eta}_S(s_2) \rangle_c &= \frac{\gamma(T_1+T_2)}{2(s_1+s_2)}, \\ \langle \tilde{\eta}_S(s_1)\tilde{\eta}_D(s_2) \rangle_c &= \frac{\gamma(T_1-T_2)}{s_1+s_2}, \\ \langle \tilde{\eta}_S(s_1)\tilde{\eta}_D(s_2)\tilde{\eta}_D(s_3)\tilde{\eta}_D(s_4) \rangle_c &= \frac{1}{\lambda} \frac{12\gamma^2(T_1+T_2)(T_1-T_2)}{s_1+s_2+s_3+s_4}.\end{aligned}\tag{C2}$$

The zero-th order on k_3 is exactly the same as the Gaussian case. The first order can be illustrative and we shall calculate it in the following. The quadratic term reads,

$$\begin{aligned}\overline{\langle j_{12} \rangle}_{12} &= k_1 \lim_{z \rightarrow 0} \lim_{\epsilon \rightarrow 0} \lim_{\alpha \rightarrow 0} \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \dots \int_{-\infty}^{\infty} \frac{dq_5}{2\pi} \frac{z(iq_2+\epsilon)}{z-(iq_1+iq_2+2\epsilon)} \frac{2k_3}{R'(iq_1+\epsilon)} \frac{\langle \tilde{r}_D(iq_3+\alpha) \tilde{r}_D(iq_4+\alpha) \tilde{r}_D(iq_5+\alpha) \tilde{r}_S(iq_2+\epsilon) \rangle_c}{iq_1+\epsilon-(iq_3+iq_4+iq_5+3\alpha)} \\ &= 24\lambda\gamma^2 k_1 k_3 \lim_{z \rightarrow 0} \lim_{\epsilon \rightarrow 0} \lim_{\alpha \rightarrow 0} \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \dots \int_{-\infty}^{\infty} \frac{dq_5}{2\pi} \frac{z}{z-(iq_1+iq_2+2\epsilon)} \frac{(iq_2+\epsilon)}{iq_1+\epsilon-(iq_3+iq_4+iq_5+3\alpha)} \times \\ &\quad \times (iq_2+\epsilon) \frac{(T_1+T_2)(T_1-T_2)}{(iq_3+iq_4+2\alpha+iq_1+iq_2+2\epsilon)} \\ &= -18\gamma^2 \frac{k_1 k_3}{\lambda} \frac{(8k^2\gamma^2 m + 20\gamma^2 k m k_1 + 6k\gamma^4 - 9\gamma^2 m k_1^2 - 4k m^2 k_1^2 - 9m^2 k_1^3)(T_1^2 - T_2^2)}{[\gamma^2(k+k_1)+m k^2][6\gamma^2(2k+3k_1)+16k^2 m + 72mk k_1 + 81m k_1^2](3\gamma^4 + 4\gamma^2 k m + 4\gamma^2 m k_1 + m^2 k_1^2)}\end{aligned}\tag{C3}$$

The quartic term reads,

$$\begin{aligned}\overline{\langle j_{12} \rangle}_{14} &= -k_3 \lim_{z \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \dots \int_{-\infty}^{\infty} \frac{dq_4}{2\pi} \frac{z(iq_2+\epsilon)}{z-(iq_1+iq_2+iq_3+iq_4+2\epsilon)} \langle \tilde{r}_D(iq_1+\epsilon) \tilde{r}_S(iq_2+\epsilon) \tilde{r}_D(iq_3+\epsilon) \tilde{r}_D(iq_4+\epsilon) \rangle_c \\ &= -12\lambda_0\gamma^2 k_3 \lim_{z \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \dots \int_{-\infty}^{\infty} \frac{dq_4}{2\pi} \frac{z}{z-(iq_1+iq_2+iq_3+iq_4+2\epsilon)} \frac{1}{R'(iq_1+\epsilon) R'(iq_3+\alpha) R'(iq_4+\epsilon) R''(iq_2+\epsilon)} \\ &\quad \times \frac{1}{R'(iq_1+\epsilon) R'(iq_3+\alpha) R'(iq_4+\alpha) R''(iq_2+\epsilon)} \frac{(T_1+T_2)(T_1-T_2)}{(iq_3+iq_4+iq_5+3\alpha+iq_2+\epsilon)} \\ &= -18\gamma^2 \frac{k_1 k_3}{\lambda} \frac{(8k^2\gamma^2 m + 20\gamma^2 k m k_1 + 6k\gamma^4 - 9\gamma^2 m k_1^2 - 4k m^2 k_1^2 - 9m^2 k_1^3)(T_1^2 - T_2^2)}{[\gamma^2(k+k_1)+m k^2][6\gamma^2(2k+3k_1)+16k^2 m + 72mk k_1 + 81m k_1^2](3\gamma^4 + 4\gamma^2 k m + 4\gamma^2 m k_1 + m^2 k_1^2)}.\end{aligned}\tag{C4}$$

The total Poisson conductivity at order 1 on k_3 reads

$$\begin{aligned}\kappa_{Poisson} &= \frac{1}{2} \frac{\gamma k^2}{\gamma^2(k+k_1)+k^2 m} + \frac{3}{2} \frac{\gamma^3 k_1 k_3 (2k+k_1)(T_1+T_2)}{(k+2k_1)[mk^2+\gamma^2(k+k_1)]^2} + \\ &\quad + 54\gamma^4 \frac{k_1 k_3}{\lambda} \frac{\mathcal{N}}{\mathcal{D}} (T_1+T_2) + \mathcal{O}(k_3^3), \\ &= \kappa^{(0)} + \kappa^{(1)} + \kappa^{(s)} + \mathcal{O}(k_3^3)\end{aligned}$$

where

$$\begin{aligned}\mathcal{N} &= 5k\gamma^2 + 3\gamma^2k_1 + 3mk_1^2 + 4k^2m + 11mk k_1 \\ \mathcal{D} &= [\gamma^2(k + k_1) + mk_1^2] [6\gamma^2(2k + 3k_1) + 16k^2m + 72mk k_1 + 81mk^2] \times \\ &\quad \times [3\gamma^4 + 4\gamma^2m(k + k_1) + m^2k_1^2].\end{aligned}\tag{C5}$$

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